Self-dual gravity with topological terms

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Abstract

The canonical analysis of the (anti-) self-dual action for gravity supplemented with the (anti-) self-dual Pontrjagin term is carried out. The effect of the topological term is to add a 'magnetic' term to the original momentum variable associated with the self-dual action leaving the Ashtekar connection unmodified. In the new variables, the Gauss constraint retains its form, while both vector and Hamiltonian constraints are modified. This shows, the contribution of the Euler and Pontrjagin terms is not the same as that coming from the term associated with the Barbero-Immirzi parameter, and thus the analogy between the θ-angle in Yang-Mills theory and the Barbero-Immirzi parameter of gravity is not appropriate.

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1 Introduction

In the first order formalism gravity supplemented with topological terms is given by the action

$$\begin{aligned} S[e,\omega] &= +\alpha_1 \int *(e^I \wedge e^J) \wedge R_{IJ}(\omega) + \alpha_2 \int e^I \wedge e^J \wedge R_{IJ}(\omega) \\ &\quad +\alpha_3 \int R^{IJ}(\omega) \wedge R_{IJ}(\omega) + \alpha_4 \int *R^{IJ}(\omega) \wedge R_{IJ}(\omega) \, . \end{aligned} \tag{1}$$

The first term in (1) is the Hilbert-Palatini action, the second one is proportional to the first Bianchi identities when there is no torsion and thus vanishes 'on shell,' namely, in a second order formalism. Third and fourth terms are the Pontrjagin and Euler terms, respectively. e^I is a non-degenerate inverse tetrad frame with Lorentz indices I, $J = 0,1,2,3,$; raised and lowered with the Lorentz metric η^{IJ}. The spacetime signature is $\eta_{IJ} =$

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diag$(-1,+1,+1,+1)$. $\omega_I{}^J$ is a Lorentz connection 1-form and $R_{IJ}(\omega)=\frac{1}{2}R_{\mu\nu IJ}(\omega)dx^\mu\wedge dx^\nu$ is its curvature, $R_{\mu\nu I}{}^J=\partial_\mu\omega_{\nu I}{}^J-\partial_\nu\omega_{\mu I}{}^J+\omega_{\mu I}{}^K\omega_{\nu K}{}^J-\omega_{\nu I}{}^K\omega_{\mu K}{}^J$. The definition of the dual operator is $*T^{IJ}=\frac{1}{2}\epsilon^{IJ}{}_{KL}T^{KL}$ with $\epsilon_{0123}=+1$.

As far as I know, the full canonical analysis of action (1) has not been carried out. When $\alpha_2=\alpha_3=\alpha_4=0$, action (1) reduces to the standard Hilbert-Palatini action whose canonical analysis is already reported in the literature [1]. There, the canonical variables are the 3-dimensional extrinsic curvature K^i_a and the densitizied inverse triad field $\widetilde{E}^a_i$ when the time gauge is chosen. On the other hand, (anti-) self-dual gravity is obtained when $\alpha_2=\epsilon i\alpha_1$, and $\alpha_3=\alpha_4=0$ with $\epsilon=+$ and $\epsilon=-$ for the self-dual and anti-self-dual actions, respectively [2, 3, 4, 5]. The canonical formalism of the (anti-) self-dual action leads to the phase space variables introduced by Ashtekar [6]. These phase space variables are a complex $SU(2)$ connection and the densitizied inverse triad field when the time gauge is fixed. The canonical formalism of the self-dual action can be carried out without fixing the gauge, which implies modifications on both the configuration variable and its momentum [7]. When $\alpha_3=\alpha_4=0$ and α_2 is a non vanishing real parameter, action (1) reduces to the action studied by Holst [8], whose canonical formalism leads to the phase space variables introduced by Barbero; which were initially found via a canonical transformation from the phase space variables associated with the Hilbert-Palatini action [9]. One of the advantages of Barbero variables is that they are real for Lorentzian gravity and their use has been crucial for the development of the quantum theory, known as loop quantum gravity [see, for instance, Ref.[10]].

The parameter α_2 gives rise to the Barbero-Immirzi parameter. Both parameters α_1 and α_2 enter in the spectra of geometric operators [11]. The usual choice is $\alpha_1=\frac{c^3}{16\pi G}$ leaving α_2 arbitrary. Up to now, Barbero-Immirzi parameter has not been fixed either by using fundamental principles nor by experimental means. It has been shown by Rovelli-Thiemann that the quantizations coming from various values for the Barbero-Immirzi parameter are inequivalent [12], which is not a particular fact of field theory and thus of gravity. In fact, the same happens even for systems with a finite number of degrees of freedom, the reason being that the group of canonical transformations is not isomorphic to the group of unitary transformations, so it is natural that systems related by canonical transformations have, in general, inequivalent quantum theories. In addition, it has been argued by Gambini-Obregón-Pullin that the Barbero-Immirzi parameter is very similar to the θ ambiguity present in Yang-Mills theories [13] in the sense that both are ambiguities of their respective quantum theories. Nevertheless, in Yang-Mills theories the term associated with the θ angle corresponds with the Pontrjagin term associated with the Faraday tensor while in gravity, as has been noted, the term in the action (1) which gives rise to the Barbero-Immirzi parameter is the second term in (1) which is related to the first Bianchi identities when there is no torsion. Therefore, the origin of both parameters, Barbero-Immirzi and θ angle, in both theories is quite different in their respective actions.

Nevertheless, the question remains, how do Euler and Pontrjagin terms in (1) contribute in the canonical formalism of general relativity? This is the issue addressed in this work. More precisely, the present canonical analysis is restricted to the (anti-) self-dual case. In

spite of this, from the present analysis it will become clear that the contribution of the second term in (1), related to the Barbero-Immirzi parameter, is completely different to the contribution of the third and fourth terms of (1). Actually, while the inclusion of the second term in (1) allows for the introduction of the Ashtekar-Barbero connection A^i_a instead of the extrinsic curvature K^i_a as the gravitational configuration variable, the effect of the third and fourth terms in (1) *does* add an extra piece to the expression for the (anti-) self-dual momentum variable, leaving the Ashtekar connection unmodified.

2 Self-dual gravity

Now follows a brief summary of the $3+1$ decomposition of the (anti-) self-dual action for completeness reasons. As already mentioned, self-dual ($\epsilon=+$) and anti-self-dual ($\epsilon=-$) gravity are obtained by setting in (1) the parameters equal to $\alpha_2=\epsilon i\alpha_1$, $\alpha_3=0$, $\alpha_4=0$, from which the (anti-) self-dual action follows:

$$S[e,\omega] \;=\; \alpha_1 \int *(e^I\wedge e^J)\wedge R_{IJ}(\omega) + \epsilon i \alpha_1 \int e^I\wedge e^J \wedge R_{IJ}(\omega)\,, \tag{2}$$

which can be rewritten as

$$S[e,{}^{(\epsilon)}\omega] \;=\; \int \left[2\epsilon i\alpha_1\, {}^{(\epsilon)}(e^I\wedge e^J)\wedge {}^{(\epsilon)}R_{IJ}\right]\,, \tag{3}$$

where ${}^{(+)}\omega$ and ${}^{(-)}\omega$ correspond with the self-dual and anti-self-dual connections, respectively. By choosing the 'time gauge,' namely, $e^0=Ndx^0$, $e^i=E^i_aN^adx^0+E^i_adx^a$ where $a=1,2,3$ denotes space indices, action (3) becomes

$$\begin{aligned} S[A^i_a,\widetilde{\Pi}^a_i,\underset{\sim}{\lambda},\lambda^a,\lambda^i] \;&=\; \int dx^0dx^3\left[\dot{A}^i_a\widetilde{\Pi}^a_i-(\underset{\sim}{\lambda}\widetilde{\widetilde{H}}+\lambda^a\widetilde{V}_a+\lambda^i\widetilde{G}_i)\right] \\ &\quad + \int dx^0\int dx^3\partial_a(\widetilde{\Pi}^a_i\lambda^i)\,, \end{aligned} \tag{4}$$

where the dependence of the phase space variables and the Lagrange multipliers on the initial Lagrangian variables is

$$\begin{aligned} \widetilde{\Pi}^a_i &:= -2\epsilon i\alpha_1 E E^a_i\,, \\ A^i_a &:= \Gamma^i_a - \epsilon i {\omega_{a\hat{0}}}^i\,, \quad \Gamma^i_a = -\frac{1}{2}\epsilon^i{}_{jk}{\omega_a}^{jk}\,, \\ \underset{\sim}{\lambda} &:= -\frac{1}{4\alpha_1}\frac{N}{E}\,, \\ \lambda^a &:= N^a\,, \\ \lambda^i &:= -A^i_0 = -(\Gamma^i_0-\epsilon i{\omega_{0\hat{0}}}^i)\,, \quad \Gamma^i_0=-\frac{1}{2}\epsilon^i{}_{jk}{\omega_0}^{jk}\,, \end{aligned} \tag{5}$$

and the first class constraints have the form

$$
\begin{aligned}
\widetilde{\widetilde{H}} &:= \epsilon^{ijk}\widetilde{\Pi}^a_i\widetilde{\Pi}^b_j F_{abk} \ , \\
\widetilde{V}_a &:= \widetilde{\Pi}^b_i F_{ab}{}^i \ , \\
\widetilde{G}_i &:= \mathcal{D}_a\widetilde{\Pi}^a_i = \partial_a\widetilde{\Pi}^a_i + \epsilon_{ij}{}^k A^j_a\widetilde{\Pi}^a_k \ ,
\end{aligned}
\tag{6}
$$

with $F_{ab}{}^i = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b$. When the 'time gauge' is not chosen, the definition of the phase space variables is modified. In particular, the momentum $\widetilde{\Pi}^a_i$ is not just proportional to the densitizied inverse triad field and Γ^i_a is not just the 3-dimensional spin connection if contact with the second order formalism is required [7].

Thus, in the Hamiltonian formulation of (anti-) self-dual gravity is present the situation found in Yang-Mills and Maxwell theories. In Maxwell theory, the Lagrangian action depends on a 4-dimensional connection A_μ and the action is fully gauge-invariant under the gauge transformation $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu\Lambda$. Once the Hamiltonian formalism of Maxwell theory is done, the configuration variable is the 3-dimensional part of A_μ, namely, A_a while the temporal part of A_μ, namely, A_0 becomes $-\lambda$ with λ the Lagrange multiplier associated with the Gauss law. The initial Lagrangian gauge symmetry $A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu\Lambda$ is, in the canonical formalism, encoded in the gauge transformation for the 3-dimensional connection $A_a \rightarrow A'_a = A_a + \partial_a\Lambda$ plus the transformation law for the Lagrange multiplier $\lambda \rightarrow \lambda' = \lambda - \dot{\Lambda}$. In (anti-) self-dual gravity, the Lagrangian action depends on the 4-dimensional connection $A^i_\mu = \Gamma^i_\mu - \epsilon i\omega_{\mu\hat{0}}{}^i$, $\Gamma^i_\mu = -\frac{1}{2}\epsilon^i{}_{jk}\omega_\mu{}^{jk}$ valued in the (complex) algebra of $SU(2)$. Once the $3+1$ decomposition of spacetime is done, the 3-dimensional part of A^i_μ becomes the configuration variable A^i_a and its temporal part A^i_0 becomes minus the Lagrange multiplier λ^i associated with the Gauss constraint. Here, as in Maxwell theory, the initial gauge symmetry present in the (anti-) self-dual action (3): 4-dimensional diffeomorphism invariance plus internal Lorentz transformations of the tetrad frame is, in the Hamiltonian formalism, encoded in the transformation law for the phase space variables and for the Lagrange multipliers [7], in particular, when spacetime $\mathcal{M} = \Sigma \times R$ and Σ has no boundary, the action (4) is fully gauge-invariant on the constraint surface under the gauge symmetry generated by the first class constraints and the Lagrange multipliers [7]. In this sense, in spite of the fact (anti-) self-dual gravity has a quadratic in the momenta first class Hamiltonian constraint, it resembles Yang-Mills and Maxwell theories which contain a Gauss constraint which is linear and homogeneous in the momenta.

Finally, in spite of the simplicity of the constraints in terms of Ashtekar variables, self-dual gravity describes complex general relativity. To recover the real sector associated with real general relativity, extra conditions must be imposed; the so-called *reality conditions*. Nevertheless, the various proposals to incorporate the reality conditions have not worked in general and the issue of how to get the real sector of self-dual gravity is still an open problem. In [14], a novel proposal to recover the real sector of self-dual gravity was proposed. There, reality conditions are implemented as second class constraints which lead to the introduction of Dirac brackets to handle them, but the price we pay for having included reality conditions as second class constraints is a non-polynomial Dirac bracket. To

avoid the use of Dirac brackets, an alternative proposal was developed in [15, 16]. There, second class class constraints are transformed into first class ones, following the method of [17]. The physical meaning of this formulation and its utility in the quantization of gravity remain unclear.

3 Self-dual gravity with topological terms

The idea is to add topological terms to (anti-) self-dual gravity. To do this, it is worth writing the action (1) in a different fashion

$$\begin{aligned} S &= \int \left[(\alpha_2 + i\alpha_1)^{(+)}(e^I \wedge e^J) \wedge {}^{(+)}R_{IJ} + (\alpha_2 - i\alpha_1)^{(-)}(e^I \wedge e^J) \wedge {}^{(-)}R_{IJ} \right] \\ &\quad + \int \left[(\alpha_3 + i\alpha_4)^{(+)}R^{IJ} \wedge {}^{(+)}R_{IJ} + (\alpha_3 - i\alpha_4)^{(-)}R^{IJ} \wedge {}^{(-)}R_{IJ} \right] . \end{aligned} \tag{7}$$

Note that when $\alpha_2 - i\alpha_1 = 0$ the second term in (7) vanishes and the first one corresponds to the self-dual action. Therefore, because just the self-dual part of the connection enters with this choice, it is natural to consider $\alpha_3 - i\alpha_4 = 0$ and also drop the fourth term which involves the anti-self-dual connection. With these choices, the remaining action is just a functional of the self-dual connection and of the tetrad frame

$$S[e, {}^{(+)}\omega] = \int \left[2i\alpha_1 \, {}^{(+)}(e^I \wedge e^J) \wedge {}^{(+)}R_{IJ} + 2i\alpha_4 {}^{(+)}R^{IJ} \wedge {}^{(+)}R_{IJ} \right] . \tag{8}$$

Last action is the self-dual action (3) supplemented with the self-dual Pontrjagin term. The anti-self-dual analog of (8) is given with $\alpha_2 + i\alpha_1 = 0$ and $\alpha_3 + i\alpha_4 = 0$. In summary, self-dual ($\epsilon = +$) and anti-self-dual ($\epsilon = -$) gravity with Pontrjagin term are given by $\alpha_2 = \epsilon i\alpha_1$, and $\alpha_3 = \epsilon i\alpha_4$. Thus, the action

$$\begin{aligned} S[e, \omega] &= +\alpha_1 \int *(e^I \wedge e^J) \wedge R_{IJ}(\omega) + \epsilon i\alpha_1 \int e^I \wedge e^J \wedge R_{IJ}(\omega) \\ &\quad + \epsilon i\alpha_4 \int R^{IJ}(\omega) \wedge R_{IJ}(\omega) + \alpha_4 \int *R^{IJ}(\omega) \wedge R_{IJ}(\omega) , \end{aligned} \tag{9}$$

which can be rewritten as

$$S[e, {}^{(\epsilon)}\omega] = \int \left[2\epsilon i\alpha_1 \, {}^{(\epsilon)}(e^I \wedge e^J) \wedge {}^{(\epsilon)}R_{IJ} + 2\epsilon i\alpha_4 {}^{(\epsilon)}R^{IJ} \wedge {}^{(\epsilon)}R_{IJ} \right] , \tag{10}$$

is the right generalization for (anti-) self-dual gravity to include topological terms. As already mentioned, both Euler and Pontrjagin terms in (9) combine to become a complex (self-dual or anti-self-dual) Pontrjagin term in (10). Now it follows the canonical formalism of (10). The 'time gauge' is chosen, namely, $e^0 = N dx^0$, $e^i = E^i_a N^a dx^0 + E^i_a dx^a$ where $a = 1, 2, 3$ denotes space indices. The action (10) becomes

$$\begin{aligned} S[A^i_a, \widetilde{\pi}^a_i, \underset{\sim}{\lambda}, \lambda^a, \lambda^i] &= \int dx^0 dx^3 \left[\dot{A}^i_a \widetilde{\pi}^a_i - (\underset{\sim}{\lambda} \widetilde{\widetilde{H}} + \lambda^a \widetilde{V}_a + \lambda^i \widetilde{G}_i) \right] \\ &\quad + \int dx^0 \int dx^3 \partial_a (\widetilde{\pi}^a_i \lambda^i) , \end{aligned} \tag{11}$$

where the dependence of the phase space variables and the Lagrange multipliers on the initial Lagrangian variables is

$$
\begin{aligned}
\widetilde{\pi}^a_i &:= \widetilde{\Pi}^a_i + 2\epsilon i\alpha_4 \widetilde{\eta}^{abc} F_{bci} = -2\epsilon i \alpha_1 E E^a_i + 2\epsilon i \alpha_4 \widetilde{\eta}^{abc} F_{bci} \,, \\
A^i_a &:= \Gamma^i_a - \epsilon i \omega_{a\hat{0}}{}^i \,, \quad \Gamma^i_a = -\frac{1}{2}\epsilon^i{}_{jk}\omega_a{}^{jk} \,, \\
\underset{\sim}{\lambda} &:= -\frac{1}{4\alpha_1}\frac{N}{E} \,, \\
\lambda^a &:= N^a \,, \\
\lambda^i &:= -A^i_0 = -(\Gamma^i_0 - \epsilon i \omega_{0\hat{0}}{}^i) \,, \quad \Gamma^i_0 = -\frac{1}{2}\epsilon^i{}_{jk}\omega_0{}^{jk} \,,
\end{aligned} \tag{12}
$$

and the first class constraints have the form

$$
\begin{aligned}
\widetilde{\widetilde{H}} &:= \epsilon^{ijk}(\widetilde{\pi}^a_i - 2\epsilon i \alpha_4 \widetilde{\eta}^{acd}F_{cdi})(\widetilde{\pi}^b_j - 2\epsilon i\alpha_4\widetilde{\eta}^{bef}F_{efj})F_{abk} \;, \\
\widetilde{V}_a &:= (\widetilde{\pi}^b_i - 2\epsilon i \alpha_4 \widetilde{\eta}^{bcd} F_{cdi}) F_{ab}{}^i \;, \\
\widetilde{G}_i &:= \mathcal{D}_a \widetilde{\pi}^a_i \;.
\end{aligned} \tag{13}
$$

Thus, the effect of the complex Pontrjagin term added to the self-dual action is as follows: i) a 'magnetic' component $2\epsilon i\alpha_4\widetilde{\eta}^{abc}F_{bci}$ is added to the standard (anti-) self-dual momentum $\widetilde{\Pi}^a_i = -2\epsilon i\alpha_1 E E^a_i$ [see Eq. (5)] to obtain the new momentum of (12), ii) the expressions of the first class constraints are modified too [see Eq. (6) and Eq. (13)], iii) finally, notice that even though α_1 (and therefore α_2) is fixed an equal to $\alpha_1 = \frac{c^3}{16\pi G}$, the parameter α_4 (and therefore α_3) is, in principle, a free parameter.

In summary, the contribution in the canonical formalism of the term proportional to the Bianchi identities when there is no torsion, which is related to the Barbero-Immirzi parameter, is completely different to the contribution coming from the topological terms. The former affects (when the time gauge is chosen) to the configuration variable while the latter affects its momentum. Therefore, Barbero-Immirzi ambiguity present in general relativity can not be, directly, identified as coming from a topological term, rather, the present analysis suggests it that the analogy between the θ-angle of Yang-Mills theory and the Barbero-Immirzi of gravity is not appropriate. It would be worth performing the canonical analysis of action (1) in two more cases: a) without fixing the gauge. At first sight, this would imply a bigger phase space with the corresponding introduction of second class constraints to recover the correct counting of degrees of freedom, b) without restricting to the (anti-) self-dual case, i.e., leaving α_2, α_3, and α_4 as arbitrary real parameters. As the present paper suggests, this fact would imply phase space variables $(A^i_a, \widetilde{\pi}^a_i)$ for the gravitational field, where $A^i_a = \Gamma^i_a + \beta K^i_a$ and $\widetilde{\pi}^a_i = \widetilde{\Pi}^a_i + \gamma\widetilde{\eta}^{abc}F_{bci}$; with $(A^i_a, \widetilde{\Pi}^a_i)$, $\widetilde{\Pi}^a_i = \frac{1}{\beta}\widetilde{E}^a_i$ the canonical pair of Barbero's formulation, i.e., two free parameters β and γ would appear in the formalism if local Lorentz symmetry is destroyed. It might be interesting to perform a Lorentz covariant canonical analysis of the action (1) too.

Finally, some words on the implications of the topological terms in the quantum theory. From the form of the constraints (13) it follows that the quantum theory might feel the topological properties of spacetime.

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